Reheating in the Presence of Inhomogeneous Noise

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Explosive particle production due to parametric resonance is a crucial feature of reheating in inflationary cosmology. Coherent oscillations of the inflaton field lead to a periodically varying mass in the evolution equation of matter and gravitational fluctuations and often induce a parametric resonance instability. In a previous paper (hep-ph/9709273) it was shown that homogeneous (i.e. space-independent) noise leads to an increase of the generalized Floquet exponent for all modes, at least if the noise is temporally uncorrelated. Here we extend the results to the physically more realistic case of spatially inhomogeneous noise. We demonstrate - modulo some mathematical fine points which are addressed in a companion paper - that the Floquet exponent is a non-decreasing function of the amplitude of the noise. We provide numerical evidence for an even stronger statement, namely that in the presence of inhomogeneous noise, the Floquet exponent of each mode is larger than the maximal Floquet exponent of the system in the absence of noise.

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I. INTRODUCTION

Explosive particle production at the end of the period of inflation is a crucial element of inflationary cosmology. It determines the effectiveness of energy transfer between the inflaton (the scalar field driving the exponential expansion of the Universe during inflation) and ordinary matter. As was first pointed out in [1] and discussed in detail in [2–5] and in many other papers (see e.g. [6]), a coherently oscillating scalar field (such as the inflaton during the period of reheating) induces a parametric resonance instability in the mode equations of bosonic matter fields to which it couples. It has recently been realized that this resonance may also amplify gravitational waves [7], scalar gravitational fluctuations [8–10], and - within the limits imposed by the exclusion principle - fermionic modes [11,12].

Resonance instabilities are, in general, quite sensitive to the presence or absence of noise. In a real physical system one expects some noise in the evolution of the scalar field driving the resonance. In the case of inflation, such noise could be due to quantum fluctuations (the same fluctuations which on larger length scales may develop into the seeds for galaxies and galaxy clusters) or to thermal excitations. In both cases, the amplitude of the noise is expected to be small. Nevertheless, it is important to study the effects of noise on the parametric resonance instability.

In a previous paper [13] we studied the effects of spatially homogeneous noise on the energy transfer between the inflaton ϕ and a second bosonic matter field χ. We were able to prove that - provided the noise is temporally uncorrelated on time scales on the order of the period of oscillation of the inflaton - the presence of noise increases the rate of energy transfer between ϕ and χ (see also [14] for similar results). More specifically, the exponential growth rate of a particular Fourier mode χ_k of χ is given by the generalized Floquet exponent μ_k. Then, if the noise is given by a space-independent function $q(t)$, the following result holds [13]:

$$\mu_k(q) \, > \, \mu_k(0) \quad \forall k \, , \tag{1}$$

if $q(t) \neq 0$. In particular, this implies that the stability bands (the bands of k values for which $\mu(k) = 0$) disappear completely.

In this paper, we extend the above result to the physically more realistic case of spatially inhomogeneous noise. This extension involves introducing nonlinearities into the physical system. Inhomogeneous noise couples different Fourier modes of χ. Hence, the mathematical analysis of the dynamics is much more complicated since the partial differential equation describing the evolution of χ can no longer be reduced via Fourier transformation to a system of decoupled ordinary differential equations. However, by introducing an ultraviolet cutoff and by considering the

system in a finite volume, we can reduce the problem to a system of coupled ordinary differential equations. We are then able to show that the methods used in [13] extend to this case. Thus, for a given cutoff theory, the result (1) can be shown to hold. When the cutoffs are removed, we obtain the slightly weaker result

$$\mu(q) \geq \mathrm{max}_k \mu_k(0) = \mu_{max} , \tag{2}$$

if $q(t,x) \neq 0$, where the Lyapunov exponent $\mu(q)$ is the maximal Lyapunov exponent of the inhomogeneous system.

By Fourier transforming the configuration $\chi(x,t)$ at each time, it is possible to define a generalized Lyapunov exponent $\mu_k(q)$ for each Fourier mode. Mode mixing induced by the inhomogeneous noise has the effect of distributing the energy which in the absence of noise would be dumped only into the original instability bands to all of the modes of χ. Hence, it is not unreasonable to expect that a much stronger result than (2) holds, namely that $\mu_k(q)$ becomes larger than the maximal Floquet index μ_{max} of the system in the absence of noise. Indeed, we find numerical evidence which supports the following conjecture:

$$\mu_k(q) \geq \mu_{max} \quad \forall k , \tag{3}$$

for a general inhomogeneous noise function $q(t,x)$.

The outline of this paper is as follows. In the following section we introduce the dynamical system to be studied and review the approach of [13] to proving (1). In Section 3, the mathematically rigorous generalization of the proof to inhomogeneous noise is outlined, relegating some technical details to a companion paper [15]. This yields a proof of (2). In Section 4 we indicate how the main result can be seen via a perturbative approach, and in Section 5 we present numerical evidence for the much stronger statement of (3).

Our results apply for both systems with narrow and broad-band resonance. For simplicity we neglect the expansion of the Universe; however, we do not believe that including effects of expansion would change our main conclusions.

II. SYSTEM WITH HOMOGENEOUS NOISE

The model we consider consists of two scalar fields ϕ and χ with an interaction Lagrangian of the form

$$\mathcal{L}_I \sim \epsilon \phi \chi^2 , \tag{4}$$

where $\epsilon \ll 1$ is a coefficient which parameterizes the strength of the interaction. In our application to inflationary cosmology, ϕ represents the inflaton field, the scalar field which drives inflation. When inflation ends, the inflaton

begins to oscillate coherently about its ground state. Superimposed on this homogeneous oscillation is quantum or thermal noise $q(t, x)$, i.e.

$$\phi(t,x) \;=\; p(\omega t) + q(t,x)\,, \tag{5}$$

for some periodic function $p(\omega t)$.

We consider $\phi(t, x)$ to be a background field. Our goal is to study the effects of the resonant excitation of χ during the period when ϕ is oscillating coherently (this period is called the "reheating" or more accurately the "preheating" period [2]). The back-reaction of the excitation of χ on the evolution of ϕ can play an important role [4–6]. However, since this is not the subject of our study, we shall neglect this effect. The equation of motion for χ is

$$\ddot{\chi} - \nabla^2\chi + \left[m_\chi^2 + (p(\omega t) + q(t,x))\right]\chi \;= 0\,, \tag{6}$$

where $p(y)$ is a function with period 2π. Both the function $p(y)$ and the noise term in (6) derive from the interaction Lagrangian (4) (and are rescaled compared to (5)).

In the absence of noise, we can diagonalize the partial differential equation (6) by Fourier transformation. The modes χ_k satisfy the following second order ordinary differential equation:

$$\ddot{\chi}_k + \left[\omega_k^2 + p(\omega t)\right]\chi_k \;= 0\,. \tag{7}$$

where $\omega_k^2 = m_\chi^2 + k^2$. This is a Mathieu or Hill equation for each k, and as is well known, there are (as a function of k) stability and instability bands. In the instability bands, χ_k grows exponentially at a characteristic rate given by the generalized Floquet index μ_k. If $p/\omega^2 \ll 1$, the instability bands are narrow, whereas in the case $p/\omega^2 \gg 1$ we have broad-band resonance. Our analysis will apply to both cases.

In the case of homogeneous noise which was analyzed in [13], the Fourier modes remain decoupled:

$$\ddot{\chi}_k + \left[\omega_k^2 + p(\omega t) + q(t)\right]\chi_k \;= 0\,. \tag{8}$$

It proves convenient to write the noise in terms of a dimensionless function $n(t)$ with characteristic amplitude 1:

$$q(t) = g\omega^2 n(t)\,, \tag{9}$$

where g is a dimensionless coupling constant which is proportional to the coupling constant in (4).

The first step in the analysis of [13] was to rewrite Equation (8) for fixed k (the index k will be dropped in the rest of this section) as a first order matrix differential equation for the fundamental solution (or transfer) matrix $\Phi_q(\tau, 0)$

$$\Phi_q(t,0) \;=\; \begin{pmatrix} \chi_1(t;q) & \chi_2(t;q) \\ \dot\chi_1(t;q) & \dot\chi_2(t;q) \end{pmatrix}\,, \tag{10}$$

consisting of two independent solutions $\chi_1(t;q)$ and $\chi_2(t;q)$ of the second order equation (8). The matrix equation reads

$$\dot\Phi_q \;=\; M(q(t),t)\Phi_q\,, \tag{11}$$

with initial conditions $\Phi_q(0,0) = I$. Here, $M(q(t),t)$ is the matrix

$$M(q(t),t) \;=\; \begin{pmatrix} 0 & 1 \\ -(\omega_k^2 + p(t) + q(t)) & 0 \end{pmatrix}\,. \tag{12}$$

For vanishing noise, i.e. $q(t) = 0$, the content of Floquet theory is that the solution of (11) can be written in the form

$$\Phi_0(t,0) \;=\; P_0(t)e^{Ct}\,, \tag{13}$$

where $P_0(t)$ is a periodic matrix function with period $T = 2\pi/\omega$, and C is a constant matrix whose spectrum in a resonance region is $spec(C) = \{\pm\mu(0)\}$ with $\mu(0) > 0$.

The noise is supposed to describe quantum or thermal fluctuations. For the derivation which follows, it is sufficient to make certain statistical assumptions about $q(t)$. We assume that the noise is drawn from some sample space Ω (which for homogeneous noise can be taken to be $\Omega = C(\Re)$), and that the noise is ergodic, i.e. the time average of the noise is equal to the expectation value of the noise over the sample space. In this case, it can be shown that the generalized Floquet exponent of the solutions of (11) is well defined by the limit

$$\mu(q) \;=\; \lim_{N\to\infty} \frac{1}{NT} \log \|\Pi_{j=1}^{N} \Phi(jT,(j-1)T)\|\,, \tag{14}$$

where $\|\cdot\|$ denotes some matrix norm (the dependence on the specific norm drops out in the large N limit). Furthermore, the growth rate $\mu(q)$ is continuous in q in an appropriate topology on Ω (see Appendix of [13] for details).

To obtain more quantitative information about $\mu(q)$ it is necessary to make further assumptions about the noise. We assume:

(i) The noise $q(t)$ is uncorrelated in time on scales larger than T, that is, $\{q(t;\kappa) \,:\, jT \le t \le (j+1)T\}$ is independent of $\{q(t;\kappa) \,:\, lT \le t \le (l+1)T\}$ for integers $l \neq j$, and is identically distributed., for all realizations κ of the noise.

(ii) Restricting the noise $q(t;\kappa)$ to the time interval $0 \le t < T$, the samples $\{q(t;\kappa) : 0 \le t < T\}$ within the support of the probability measure fill a neighborhood, in $C(0,T)$, of the origin.

Hypothesis (i) implies that the noise is ergodic, and therefore the generalized Floquet exponent is well defined. The main result which was proved in [13] is that $\mu(q)$ is strictly larger than $\mu(0)$;

$$\mu(q) \;>\; \mu(0)\,, \tag{15}$$

which demonstrates that the presence of noise leads to a strict increase in the rate of particle production. The proof was based on an application of Furstenberg's theorem, which concerns the Lyapunov exponent of products of independent identically distributed random matrices $\{\Psi_j : j = 1, ..., N\}$.

Consider a probability distribution dA on the matrices $\Psi \in \mathrm{SL}(2n, \mathrm{R})$. Let G_A be the smallest subgroup of $\mathrm{SL}(2n, \mathrm{R})$ containing the support of dA.

Theorem 1: *(Furstenberg, [16]) Suppose that G_A is not compact, and that the action of G_A on the set of lines in $\mathrm{R}^{2\mathrm{n}}$ has no invariant measure. Then for almost all independent random sequences $\{\Psi_j\}_{j=1}^{\infty} \subseteq \mathrm{SL}(2n, \mathrm{R})$ with common distribution dA,*

$$\lim_{N\to\infty} \frac{1}{N} \log(\|\Pi_{j=1}^{N} \Psi_j\|) = \lambda > 0\ .$$

Furthermore, for given $v_1, v_2 \in \mathrm{R}^{2\mathrm{n}}$, then

$$\lim_{N\to\infty} \frac{1}{N} \log(\langle v_1, \Pi_{j=1}^{N} \Psi_j v_2\rangle) = \lambda$$

for almost every sequence $\{\Psi_j\}_{j=1}^{\infty}$.

In order to apply Furstenberg's theorem to obtain (15), we start by factoring out from the transfer matrix $\Phi_q(t,0)$ the contribution due to the evolution without noise;

$$\Phi_q(t,0) \;=\; \Phi_0(t,0)\Psi_q(t,0) \;=\; P_0(t)e^{Ct}\Psi_q(t,0)\,. \tag{16}$$

The reduced transfer matrix $\Psi_q(t,0)$ satisfies the following equation

$$\dot{\Psi}_q \;=\; S(t;\kappa)\Psi_q \;=\; \Phi_0^{-1}(t)\begin{pmatrix} 0 & 0 \\ -q(t;\kappa) & 0 \end{pmatrix}\Phi_0(t)\Psi_q\,, \tag{17}$$

which can be written as a matrix integral equation

$$\Psi_q(t,0) \;=\; I + \int_0^t d\tau \; S(\tau;\kappa)\Psi_q(\tau,0)\,. \tag{18}$$

By properties (i) of decorrelation of the noise, the quantities $\Psi_q(jT,(j-1)T)$ are independent and identically distributed for different integers j, and we can apply the Furstenberg theorem to the following decomposition of $\Psi_q(NT,0)$;

$$\Psi_q(NT,0) \;=\; \Pi_{j=1}^{N}\Psi_q(jT,(j-1)T)\,. \tag{19}$$

We may choose for instance the vector v_1 in Theorem 1 to be an eigenvector of $\Phi_0(T,0)^t = (P_0(T)e^{CT})^t$, the transpose of the transfer matrix of the system without noise, with eigenvalue $e^{\mu(0)T}$. Then the second statement of Theorem 1 becomes

$$\begin{aligned}
\frac{1}{NT}\log|\langle v_1, \Phi_q(NT,0)v_2\rangle| &= \frac{1}{NT}\log|\langle v_1, P_0(NT)e^{CNT}\Psi_q(NT,0)v_2\rangle| \\
&= \frac{1}{NT}\log(e^{\mu(0)NT}|\langle v_1, \Psi_q(NT,0)v_2\rangle|) \\
&= \mu(0) \;+\; \frac{1}{NT}\log|\langle v_1, \Pi_{j=1}^{N}\Psi_q(jT,(j-1)T)v_2\rangle|\,.
\end{aligned} \tag{20}$$

Taking the limit $N\to\infty$ and applying the first and second statements of Theorem 1 we obtain

$$\mu(q) \;=\; \mu(0)+\lambda \;>\; \mu(0) \tag{21}$$

which proves the main result (15). Note in particular that (15) implies that the stability bands of the system without noise disappear when spatially homogeneous noise is added.

III. EXACT RESULTS

In this section we will generalize the results of [13] to the case of inhomogeneous noise, modulo some technical points which will be addressed in a separate paper [15]. To keep the notation simple, we will take space to be one-dimensional. However, the analysis works in any spatial dimension. The starting point is the second order differential equation (6) which we can rewrite as a first order matrix operator equation

$$\dot\Phi_q \;=\; M(q(t,\cdot),t)\Phi_q \tag{22}$$

with initial condition $\Phi_q(0)=I$ for the fundamental solution matrix $\Phi_q(t)$ with kernel $\Phi_q(t;x,y)$ given by

$$\Phi_q(t;x,y) \;=\; \begin{pmatrix} \chi_1(t;x,y) & \chi_2(t;x,y) \\ \dot\chi_1(t;x,y) & \dot\chi_2(t;x,y) \end{pmatrix}\,, \tag{23}$$

consisting of two independent kernels $\chi_1(t;x,y)$ and $\chi_2(t;x,y)$ of the linear operator equation (6). Here, $M(q(t,x),t)$ is the matrix operator

$$M(q(t,x),t) \;=\; \begin{pmatrix} 0 & 1 \\ -(-\nabla^2 + m_\chi^2 + p(t) + q(t,x)) & 0 \end{pmatrix} . \tag{24}$$

The operators are taken to act on the Hilbert space $H = H^1(\mathrm{R}^3) \times L^2(\mathrm{R}^3)$, where H^1 stands for the Sobolev space of L^2 functions with L^2 gradients. The first factor refers to the coordinate function, the second to the velocities. Physically, our choice of function space corresponds to considering finite energy configurations.

As in the previous section, we introduce rotating coordinates in order to factor out the time evolution in the absence of noise:

$$\Phi_q(t) \;=\; \Psi_q(t)\Phi_0(t)\,, \tag{25}$$

where $\Phi_0(t)$ is the fundamental solution matrix operator in the absence of noise, and the product implies operator composition. The initial conditions are $\Psi_q(0) = I$. In Fourier space this is a generalization of the matrix given in the previous section (13). The kernel of the matrix operator $\Psi_q(t)$ satisfies the reduced differential equation

$$\dot{\Psi}_q(t;x,y) \;=\; \int S(t;x,z_1;q)\Psi_q(t;z_1,y)dz_1 \;=\; \int\int \Phi_0^{-1}(t;x,z_2) \begin{pmatrix} 0 & 0 \\ -q(t,z_2) & 0 \end{pmatrix} \Phi_0(t;z_2,z_1)\Psi_q(t;z_1,y)dz_1dz_2\,. \tag{26}$$

The corresponding operator equation can also be written as an integral equation

$$\Psi_q(t) \;=\; I + \int_0^t dt' S(t';q)\Psi_q(t')\,. \tag{27}$$

We will focus on the reduced transfer matrix operator

$$\Psi_q(NT) \;=\; \Pi_{j=1}^N \Psi_q(jT,(j-1)T) \tag{28}$$

which describes the evolution from $t = 0$ to $t = NT$, where T is the period of the oscillating source $p(t)$, factored into matrix operators $\Psi_q(jT,(j-1)T)$ which describe the evolution in each individual period. By differentiating (27) with respect to the noise function q we obtain the action of the transfer matrix operator in the Lie algebra of infinitesimal displacements $r(t,x)$

$$\delta\Psi_q(T)\cdot r = \int_0^T \; dt \;\; \Phi_0^{-1}(t) \begin{pmatrix} 0 & 0 \\ -r(t,\cdot) & 0 \end{pmatrix} \Phi_0(t) \;. \tag{29}$$

This construction will be used in the derivation of our main results.

The first relevant mathematical result is that the growth rate (or generalized Floquet or Lyapunov exponent) of the solutions of (22) is well defined by the limit

$$\mu(q) \;=\; \lim_{N\to\infty} \frac{1}{NT} \log \|\Pi_{j=1}^{N} \Phi(jT,(j-1)T)\| \,, \tag{30}$$

where $\|\cdot\|$ denotes the operator matrix norm

$$\|\Phi\| \;=\; sup_{v\in H^1\times L^2} \|\Phi(v)\|_H \,, \tag{31}$$

where the subscript H indicates that the function norm in the Hilbert space H is used. Furthermore, the growth rate $\mu(q)$ is continuous in q in an appropriate topology on the noise sample space. Both of these results were already demonstrated in the Appendix of [13] to which the reader is referred for details.

The Lyapunov exponent of the wave equation without noise, i.e. of the wave operator $\Phi_0(t)$ is given by

$$\mu_0 \;=\; \lim_{t\to\infty} \frac{1}{t} \log \|\Phi_0(t)\| \;=\; max_{k\in R^3} \mu_k(0) \,, \tag{32}$$

where $\mu_k(0)$ is the Lyapunov exponent of the k'th mode of the system without noise.

We assume that the noise function $q(t,x)$ is given by a stochastic process which is temporally uncorrelated over times t larger than the period T. In this case, the quantities $\Psi_q(jT,(j-1)T)$ are identically distributed uncorrelated random matrix operators. The main theorem of this paper is the following:

Theorem 2: When a spatially inhomogeneous noise $q(t,x;\omega)$ is introduced into the problem, then the Lyapunov exponent can only increase, i.e.

$$\mu(q) \;\geq\; \mu_0$$

for almost all realization ω of the noise.

The main idea of the proof is to reduce the infinite dimensional operator problem to a finite dimensional matrix problem to which Furstenberg's Theorem in the form stated in Section 2 can be applied. First, we introduce an infrared cutoff by putting the system in a box. This corresponds to a discretization of the Fourier modes. The separation of k values is denoted by Δk. Next, we impose an ultraviolet cutoff in momentum space by eliminating all modes with frequency larger than k_{max}. This reduces the problem to a finite dimensional case to which Furstenberg's Theorem applies. We will prove that the hypotheses of the theorem are indeed satisfied.

In order to be able to take the limit when the ultraviolet cutoff goes to infinity and the infrared cutoff to zero, it is important to prove that the transfer matrix Ψ_q in the continuum system is a compact operator perturbation of the transfer matrix operator without noise. This implies that the perturbation can be well approximated by the corresponding transfer matrix operators of the finite dimensional approximate problem obtained after imposing the cutoffs. Since a product of compact operators is compact, it is sufficient to prove that both $\Phi_0(T,x) - \Phi_0(T,x)_{|p=0}$ and $\Psi_q(T,x) - I$ are compact independently (the first term describes the difference in the transfer matrix when the periodic part of the perturbation, namely $p(t)$, is turned on, the second term expresses the effect of non-vanishing $q(x,t)$).

Theorem 3: The operators $\Phi_0(T,x) - \Phi_0(T,x)_{|p=0}$ and $\Psi_q(T,x) - I$ are compact.

The compactness of $\Phi_0(T,x) - \Phi_0(T,x)_{|p=0}$ follows via the known asymptotics of the eigenfunctions of Sturm-Liouville operators. The compactness of $\Psi_q(T,x) - I$ can be shown explicitly [15].

Given the infrared and ultraviolet cutoffs introduced above, it is natural to work in Fourier space. In this basis, the partial differential equation (6) for $\chi(t,x)$ becomes a system of $2n$ coupled ordinary differential equations for the Fourier components $\chi_k(t)$ and their momenta $\dot{\chi}_k(t)$, where n is the number of independent Fourier modes. In other words, the transfer matrix equation (22) becomes a $2n$ dimensional vector equation with a $2n \times 2n$ transfer matrix $\Phi_q(t)$. If we order the basis vectors such that

$$\chi = \begin{pmatrix} \chi_{k_1} \\ \dot{\chi}_{k_1} \\ \cdots \\ \chi_{k_n} \\ \dot{\chi}_{k_n} \end{pmatrix}, \tag{33}$$

where $k_1, ..., k_n$ label the finite set of momenta we are considering, then $\Phi_0(t,x)$, the transfer matrix in the absence of noise, is block diagonal

$$\Phi_0(t,x) = \begin{pmatrix} \Phi_{0,k_1} & 0 & 0 \\ 0 & \cdots & \\ 0 & 0 & \Phi_{0,k_n} \end{pmatrix}. \tag{34}$$

Here, the matrices Φ_{0,k_i} are the 2×2 matrices for the k_i'th mode in the absence of noise (see Section 2). The matrix S of (26), on the other hand, mixes the blocks. If we define a new matrix Q by

$$S(t,x;q) = \Phi_0(t,x)^{-1} Q(t,x;q) \Phi_0(t,x) \tag{35}$$

and denote the Fourier transform of $q(t,x)$ by $\tilde{q}(t,k)$, then

$$Q(t,x;q) \;=\; \begin{pmatrix} Q_{11} & Q_{12} & .. & Q_{1n} \\ Q_{21} & Q_{22} & .. & Q_{2n} \\ .. & .. & .. & .. \\ Q_{n1} & Q_{n2} & .. & Q_{nn} \end{pmatrix}, \tag{36}$$

where Q_{ij} is the 2×2 matrix

$$Q_{ij}(t, k_i - k_j) \;=\; \begin{pmatrix} 0 & 0 \\ \tilde{q}(t, k_i - k_j) & 0 \end{pmatrix}. \tag{37}$$

We are now ready to sketch the proof of the main theorem. It is sufficient to show that the hypotheses of the Furstenberg Theorem (see Section 2) are satisfied. Let us denote

$$\Psi_j \;=\; \Psi_q(jT, (j-1)T) \quad \in SL(2n, R). \tag{38}$$

and consider G_A, the smallest subgroup of $SL(2n, R)$ containing the support of dA, the set of transfer matrices which can be obtained from realizations of the noise drawn from the noise sample space. We must show that

(i) G_A is not compact.

(ii) G_A acting on the set of lines in R^{2n} has no invariant measure, i.e. there is no subspace of R^{2n} left invariant under the action of all of the transfer matrices corresponding to noise drawn from the sample space.

To show that G_A is not compact, it is sufficient to consider the subset $H_A \subset R^{2n}$ corresponding to homogeneous noise and to show that H_A is not compact. Thus, we take

$$\tilde{q}(t, k_i - k_j) \;=\; q(t)\delta(k_i - k_j). \tag{39}$$

Thus, H_A is block diagonal and does not mix the basis vectors corresponding to different values of k. Since the set $\{q(t)\}$ is not compact, it follows that within each block, H_A is non-compact.

Turning to the second hypothesis of Furstenberg's Theorem, it follows from our previous study of homogeneous noise (see Appendix of [13]) that no subspace of R^{2n} corresponding to a fixed value of k is invariant under the action of H_A and hence of G_A. To show that there cannot be an invariant subspace which is different from a subspace of an R^2 associated with an individual value of k, we argue by contradiction. It is convenient to do the analysis at the level of the Lie algebra associated with the group of transfer matrices, namely the set of infinitesimal displacements given by (29), and then to use the implicit function theorem to recover the corresponding result at the group level.

Assuming that there exists an invariant subspace $X \subset R^{2n}$, we construct a special noise function $q(t,x)$ such that the associated transfer matrix takes some vector $v \in X$ into a vector $w \notin X$. For example, let us assume that the

subspace X contains a vector $v \in R^2_\alpha$ associated with the α'th Fourier mode, but that it is orthogonal to a vector $w \in R^2_\beta$ associated with the β'th Fourier mode. Consider now a noise function which is defined by

$$\tilde{q}(t, k_i - k_j) \,=\, q(t)\delta(k_i - k_\alpha)\delta(k_j - k_\beta)\,. \tag{40}$$

Under this noise function, the vector v is mapped onto a vector v' which is not orthogonal to w. The generalization of this argument to arbitrary vectors v and w will be given in [15].

IV. BORN APPROXIMATION

In this section we follow our previous work [13] closely. The inflaton field is taken to be a real scalar field and is denoted by ϕ. In the period immediately after inflation, $\phi(t)$ is assumed to be oscillating coherently about (one of) its ground state(s). As in the previous section, we shall, however, include a small aperiodic, inhomogeneous perturbation (cf. eq. (5)).

We shall (following [1–3]) take ϕ to be coupled to a second scalar field χ which represents matter via an interaction Lagrangian which is quadratic in χ, for example of the form (4). To simplify the discussion, we neglect nonlinearities in the equation of motion for χ. Such nonlinearities may be important and lead to an early termination of parametric resonance. This topic has recently been discussed extensively, see e.g. [4,6], but is not the topic of our paper. Instead, we are interested whether the presence of noise such as included in Eq. (5) will affect the onset of resonance.

For simplicity, we will neglect the expansion of the Universe. In [1–3] it has been shown that it is possible to include the expansion without any problems and that it does not affect the results concerning the parametric resonance instability. Since the equation of motion for $\chi(x,t)$ is linear, we can write down the evolution equation for the Fourier modes of χ, denoted by $\chi_k(t)$ or, equivalently, by $\chi(\mathbf{k},t)$. The equation of motion for the χ field is given by (6), where $q(t,\mathbf{x})$ represents the noise which we consider as a perturbation of the driving function $p(t)$. Taking the Fourier Transform of equation (6) and using the convolution theorem we obtain the equation of evolution for the k-mode

$$\ddot{\chi}_k + \left[\omega_k^2 + p(\omega t)\right]\chi_k = -\int d^3\mathbf{k}'\, q(\mathbf{k}' - \mathbf{k}, t)\chi(\mathbf{k}', t) \tag{41}$$

where $\omega_k^2 = m_\chi^2 + \mathbf{k}^2$, and $\chi_k \equiv \chi(\mathbf{k},t)$ is the Fourier transform of $\chi(\mathbf{x},t)$.

Note that all modes on the right hand side of Eq. (41) will influence the time evolution of the particular k-mode we are considering. This is a very important aspect of our analysis for the case of inhomogeneous noise.

A straightforward calculation shows that (41) is equivalent to an integral equation (resembling Volterra's equation), namely

$$\chi_k(t) = \chi_k^{(h)} + \chi_k^{(1)} = \chi_k^{(h)} - \int_{t_i}^{t} dt' G(t,t') \int d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)\chi(\mathbf{k}',t) \tag{42}$$

where t_i is the time at the beginning of reheating and $\chi_k^{(h)}$ is the general solution of the "homogeneous equation"

$$\ddot{\chi} + \left[\mathbf{k}^2 + m_\chi^2 + p(\omega t)\right]\chi = 0. \tag{43}$$

For the particular case when $q(\mathbf{k}'-\mathbf{k},t) = q(t)\delta(\mathbf{k}'-\mathbf{k})$ we obtain the equation discussed in [13].

Since $p(\omega t)$ is a periodic function, $\chi_k^{(h)}$ can be written in the Floquet form

$$\chi_k^{(h)} = e^{\mu_k t}p_1(t) + e^{-\mu_k t}p_2(t) \tag{44}$$

where $p_1(t)$ and $p_2(t)$ are periodic functions (which include the arbitrary constants determined by initial conditions) and μ_k is the Floquet exponent.

The solutions $\chi_1(t) = e^{\mu_k t}p_1(t)$ and $\chi_2 = e^{-\mu_k t}p_2(t)$ are linearly independent and we can choose them in such a way that their Wronskian $W = \chi_1(t)\dot{\chi_2}(t) - \dot{\chi_1}(t)\chi_2(t)$ is time independent (see e.g. [17]). Then, the kernel $G(t,t')$ in Eq.(42) is given by

$$G(t,t') = \frac{\chi_1(t')\chi_2(t) - \chi_1(t)\chi_2(t')}{W} \tag{45}$$

Our method, in this section, consists of finding an approximate solution of the integral equation (42), by reducing it as closely as possible to the equation in the case of homogeneous noise previously obtained in [13]. In the first approximation we replace $\chi(\mathbf{k}',t)$ in the momentum-space integral in (42) by the homogeneous solution $\chi^{(h)}(\mathbf{k}',t)$. The integral then reduces to

$$\int d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)p_1(\mathbf{k}',t)e^{\mu_k t} + \int d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)p_2(\mathbf{k}',t)e^{-\mu_k t} \tag{46}$$

If we restrict to the case of narrow resonance, the main contribution to the above integral comes from the first resonance band, since higher bands are of the order $o(\epsilon^2)$ or higher. In this case the integrals in the above expression can be replaced by integrals over a finite domain, namely, the first resonance band which, in momentum space, is given by the shell $k_1 \le |\mathbf{k}'| \le k_2$. For our analysis we do not need to know the values of k_1 and k_2. The expression (46) above reduces to

$$\int_{|\mathbf{k}|=k_1}^{|\mathbf{k}|=k_2} d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)p_1(\mathbf{k}',t)e^{\mu_k t} + \int_{|\mathbf{k}|=k_1}^{|\mathbf{k}|=k_2} d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)p_2(\mathbf{k}',t)e^{-\mu_k t} \tag{47}$$

If we assume that the periodic function p_2 has an amplitude of the same order as (or less than) the function p_1, then we can neglect the integral containing p_2, since its integrand is exponentially decreasing. Thus we are left with only the first integral in (47). Now, for this integral, we apply the following Mean Value Theorem [18]:

Theorem (Mean Value in 3 Dim): Let $f, g : V \subset \mathcal{R}^3 \longrightarrow \mathcal{R}$ be continuous functions, where V is a compact domain in $\mathcal{R}^3$. Then there exists a $\overline{x} \in V$ such that

$$\int_V f(\mathbf{x})g(\mathbf{x})d^3\mathbf{x} = f(\overline{\mathbf{x}}) \int_V g(\mathbf{x})d^3\mathbf{x}\,. \tag{48}$$

Using this theorem, (47) reduces to

$$e^{\mu_{\overline{k}} t} \int_{|\mathbf{k}|=k_1}^{|\mathbf{k}|=k_2} d^3\mathbf{k}' q(\mathbf{k}'-\mathbf{k},t)p_1(\mathbf{k}',t) \equiv e^{\mu_{\overline{k}} t} A(\mathbf{k},t)\,, \tag{49}$$

where $\overline{\mathbf{k}}$ is inside the resonance band. This is our final expression which estimates the contribution of all modes in the evolution equation of the mode $\mathbf{k}$. So, using the above result, our earlier evolution equation (41) reads

$$\ddot{\chi}_k + \left[\omega_k^2 + (p(\omega t)\right] \chi_k = -A(\mathbf{k},t)e^{\mu_{\overline{k}} t} \tag{50}$$

A closer analysis of the function $A(\mathbf{k},t)$, defined in (49), shows that it is bounded. This is because p_1 is periodic and the noise is by assumption a random function with a small amplitude which is identically distributed over time periods of T. Thus, we can use the approximation

$$|A(\mathbf{k},t)| \leq M_1 \sigma_k \omega^3 \epsilon\,, \tag{51}$$

where M_1 and σ_k are, respectively, upper bounds of the functions $p_1(\mathbf{k}',t)$ and $q(\mathbf{k}'-\mathbf{k},t)$ as both variables $\mathbf{k}'$ and t are varied. The last two factors in (51) represent an estimate of the volume of the first (and dominant) instability band. Also, let M_2 be the upper bound on $p_2(\mathbf{k}',t)$.

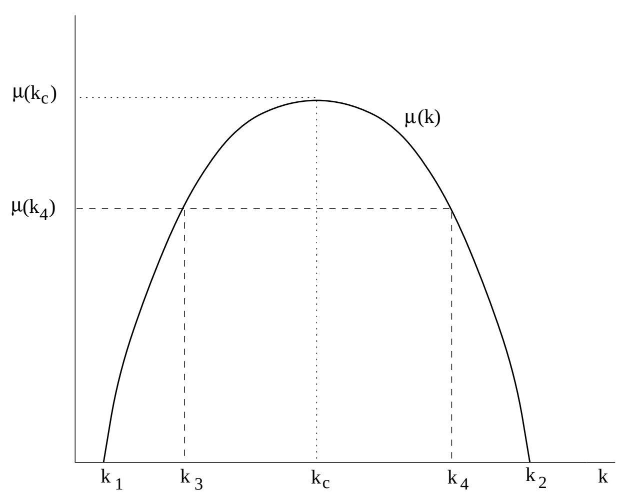

FIG. 1. The Floquet exponent μ is plotted (vertical axis) against k/ω (horizontal axis), for a general case of the Hill equation

In the resonance band $[k_1, k_2]$, we must consider the Floquet exponent as a function of k. The exact formula for μ_k can be complicated, but it can be approximated very well by a circle of arc centered at the midpoint of the interval $[k_1, k_2]$. Let k_c denote the central mode with the maximum exponent. Note that $\mu(k_1) = \mu(k_2) = 0$ (see Fig. 1). Then, if $\overline{k} \neq k_c$, we have $\mu_{\overline{k}} \leq \mu_k$, for all k satisfying $|k - k_c| \leq |\overline{k} - k_c|$. We will now consider the subinterval $[k_3, k_4] \subset [k_1, k_2]$ "centered" at k_c for which this is the case, and we shall show that for values of k in this subinterval the noise does not destroy the parametric resonance.

From Eqs. (42) and (49) it follows that

$$\chi_k^{(1)}(t) = -\int_{t_i}^{t} dt' G(t, t') A(\mathbf{k}, t') e^{\mu_{\overline{k}} t'} \tag{52}$$

where the Green's kernel $G(t, t')$ is given by (45). Using (51) it can easily be shown that

$$|\chi_k^{(1)}| \leq \sigma_k f(k, \Delta t) e^{\mu_k \Delta t} \tag{53}$$

where $\Delta t = t - t_i$ and the function $f(k, \Delta t)$ is given by

$$f(\mathbf{k}, \Delta t) = M_1^2 M_2 W^{-1} e^{\mu_k t_i} (1 - e^{-\mu_k \Delta t}) \epsilon \omega^3 \frac{1}{\mu_k} \tag{54}$$

Comparing (53) with the growth rate of the homogeneous solution $\chi_k^{(h)}$, that is $e^{\mu_k \Delta t}$, we find

$$\frac{|\chi_k^{(1)}|}{e^{\mu_k \Delta t}} \leq \sigma_k f(k, \Delta t) \tag{55}$$

for all $\mathbf{k}$ in the subinterval $k_3 \leq |\mathbf{k}| \leq k_4$ inside the resonance band (a shell in the 3-dim momentum space) $[k_1, k_2]$.

The above inequality shows that small amplitude noise (which can be controlled by the parameter σ_k) will not change the exponential rate of growth in the parametric resonance regime.

This method applies only under very restrictive hypotheses such as the assumption of narrow resonance band dynamics, applicability of the First Born Approximation method, and yields only a result in a subinterval of the resonance band which may be much narrower than the band itself (in fact, for $\overline{k} = k_c$, only for the central mode k_c we could guarantee that noise does not eliminate the exponential growth). However, based on the numerical results to be presented in the following section we gain some confidence that the methods could be generalized to less restrictive hypotheses (perhaps including the case of broad resonance). This will be the subject of further studies. A very important aspect in our analysis is the fact that the noise considered here is quite general. This is good since there are many possible sources of noise during and before the end of the inflationary period.

V. NUMERICAL RESULTS

In order to estimate the effects of inhomogeneous noise on the resonant field through numerical approximations we take the "discrete" Fourier Transform of equation (6) to get

$$\ddot{\chi}_{\mathbf{k}}(t) + \left[\omega_k^2 + p(\omega t)\right] \chi_{\mathbf{k}}(t) = -\sum_{\mathbf{k}'} q_{\mathbf{k}'-\mathbf{k}}(t)\chi_{\mathbf{k}'}(t)\,, \tag{56}$$

where $\chi(\mathbf{x},t) = \sum_{\mathbf{k}} \chi_{\mathbf{k}}(t) \exp[i\mathbf{k}.\mathbf{x}]$, etc.

Let us analyze the time evolution of a given mode $\mathbf{k}$. We can write (56) in the following form

$$\ddot{\chi}_{\mathbf{k}}(t) + \left[\omega_k^2 + p(\omega t) + q_o(t)\right] \chi_{\mathbf{k}}(t) = F_{\mathbf{k}}\,, \tag{57}$$

where $F_{\mathbf{k}} \equiv -\sum_{\mathbf{k}'\neq\mathbf{k}} q_{\mathbf{k}'-\mathbf{k}}(t)\chi_{\mathbf{k}'}(t)$ does not contain $\chi_{\mathbf{k}}$ and then may be thought of as an external force.

The general solution to (57) has the form

$$\chi_{\mathbf{k}} = \chi_{\mathbf{k}}^h + \chi_{\mathbf{k}}^p\,, \tag{58}$$

where $\chi_{\mathbf{k}}^h$ is the solution to the homogeneous equation

$$\ddot{\chi}_{\mathbf{k}}(t) + \left[\omega_k^2 + p(\omega t) + q_o(t)\right] \chi_{\mathbf{k}}(t) = 0\,, \tag{59}$$

and $\chi_{\mathbf{k}}^p$ is the particular solution of (57)

$$\chi_{\mathbf{k}}^p(t) = \int_{t_o}^{t} G(t,t')F_{\mathbf{k}}(t')dt'\,. \tag{60}$$

$G(t,t')$ is the Green function associated to (59)

$$G(t,t') = \frac{\chi_1(t')\chi_2(t) - \chi_1(t)\chi_2(t')}{W}\,, \tag{61}$$

where χ_1 and χ_2 are the two independent solutions of (59) and W is the Wronskian, which in the present case is a constant.

Equation (59) is the same as the evolution equation for the $\mathbf{k}$-mode of the boson field coupled to the inflaton during the pre-heating phase of the Universe and in the presence of homogeneous noise. This problem was studied at length in [13]. According to the results obtained in that analysis, the presence of ergodic noise strictly increases the rate of particle production. In other words, the generalized Floquet exponent in the presence of noise $\mu_k(q_o)$ is greater than $\mu_k(0)$, the Floquet exponent for $q_o(t) = 0$. We can write $\mu_k(q_o) = \mu_k(0) + \lambda_k(q_o)$ with $\lambda_k(q_o) > 0$. We then have two

distinct situations to analyze depending upon whether the **k** mode is in a resonance band or not. Modes in a stability band have $\mu_k(0) = 0$, while the modes in a resonance band have $\mu_k(0) > 0$.

To simplify the discussion we neglect the contribution of $q_o(t)$, i.e. set $\lambda_k(q_o) = 0$ in Eq. (59). This assumption will not change the qualitative results derived below. Moreover, for the situations we are interested in, where the noise is in general small when compared to the amplitude of the inflaton, and for modes close to the center of the resonance band, $\lambda_k(q_o)$ is small compared to $\mu_k(0)$ and can be neglected.

Let us first assume that **k** is in a resonance band. Then according to the theory of the Hill equation, the solution to (59) with $q_o(t) = 0$ is

$$\chi^h_{\mathbf{k}}(t) = e^{\mu_k(0)t} p_1(t) + e^{-\mu_k(0)t} p_2(t) \tag{62}$$

where p_1 and p_2 are bounded (periodic) functions of time. The solution to (57) for the modes within any resonance band and for late times can be written as

$$\chi_{\mathbf{k}}(t) \simeq e^{\mu_k(0)t} p_1(t) + \chi^p_{\mathbf{k}}(t)\,. \tag{63}$$

From the above result we see that $\chi_{\mathbf{k}}(t)$ has an exponentially growing component independent of $\chi^p_{\mathbf{k}}(t)$. Therefore, the exponential growth of a mode in a resonance band can only be destroyed by inhomogeneous noise if there is an exact cancellation between $\chi^h_{\mathbf{k}}(t)$ and $\chi^p_{\mathbf{k}}(t)$. This is only possible if $\chi^p_{\mathbf{k}}(t)$ is given by (at least for late times) $-e^{\mu_k(0)t} p_1(t)$, which is very unlikely since the initial conditions for all the modes are involved (see below). Notice that $\chi^p_{\mathbf{k}}(t)$ can in fact grow exponentially in time and reinforce the increase of $\chi_k(t)$, since all modes are coupled through the term on the right hand side of (56). The result is a stimulated resonance effect on the particular components, even for modes in a stability band (see below).

We now study the time evolution of non-resonant modes. Our goal is to estimate the rate of growth of such modes. For the sake of simplicity, we take $q_o(t) \simeq 0$ as above. Therefore, the homogeneous solution $\chi^h_{\mathbf{k}}(t)$ is given by two independent periodic functions $\chi_1(t)$ and $\chi_2(t)$. As we are interested in the exponentially growing components, we are left with only the particular solution $\chi^p_{\mathbf{k}}(t)$ to compute. In the following we proceed to find an upper bound for this "particular" solution. The results hold for all modes in a stability band. In order to estimate this solution, which is given by (60), we first need an approximation for $F_{\mathbf{k}}(t)$. Let us then explicitly determine the contribution from the homogeneous and particular solutions of each mode. Using (58) and the definition of $F_{\mathbf{k}}(t)$, we can write

$$F_{\mathbf{k}}(t) = -\sum_{\mathbf{k}'\neq\mathbf{k}} q_{\mathbf{k}'-\mathbf{k}}(t)\left(e^{\mu_{k'}(0)t}P_{\mathbf{k}'}(t) + e^{-\mu_{k'}(0)t}Q_{\mathbf{k}'}(t) + \chi^p_{\mathbf{k}'}(t)\right), \tag{64}$$

where $P_{\mathbf{k}}(t)$ and $Q_{\mathbf{k}}(t)$ are periodic functions of t. Let us recall that $\mu_k(0)$, the Floquet exponent, is (according to our definition) zero for modes in a stability band and positive for resonant modes. To begin with, we neglect bounded and decaying components in the above equation and keep just exponentially growing terms. Thus, using (61) and recalling that in the present case the solution χ_1 and χ_2 that enter the Green's function $G(t,t')$ are periodic functions of t, we can write

$$\begin{aligned}\chi^p_{\mathbf{k}}(t) = &-\sum_{\mathbf{k}'\in B}\int_{t_o}^{t} G(t,t')q_{\mathbf{k}'-\mathbf{k}}(t')\,e^{\mu_{k'}(0)t'}P'_{\mathbf{k}}(t) - \sum_{\mathbf{k}'\neq\mathbf{k}}\int_{t_o}^{t} G(t,t')q_{\mathbf{k}'-\mathbf{k}}(t')\chi^p_{\mathbf{k}'}(t') \\ &+\text{terms not growing exponentially in time}, \end{aligned} \tag{65}$$

where B stands for the sub-sets of the $\mathbf{k}$-space $\mathbf{R^3}$, such that $\mathbf{k}$ is in a resonance band. Continuing our approximations, we consider the fastest growing term on the right had side of (65). Let k_c denote such a mode so we can cast (65) in the form

$$\chi^p_{\mathbf{k}}(t) = -\int_{t_o}^{t} G(t,t')q_{\mathbf{k}_c-\mathbf{k}}(t')\,e^{\mu_{k_c}(0)t'}P_{\mathbf{k}_c}(t) - \sum_{\mathbf{k}'\neq\mathbf{k}}\int_{t_o}^{t} G(t,t')q_{\mathbf{k}'-\mathbf{k}}(t')\chi^p_{\mathbf{k}'}(t') + R, \tag{66}$$

where R stands for terms that grow slower than $e^{\mu_{k_c}(0)t}$. The first term in the above equation guarantees an exponential growth of $\chi^p_{\mathbf{k}}(t)$ with the same rate as the fastest growing resonant mode. It is also worth noting that this leading term does not depend on the initial condition of the mode itself, but just on the initial conditions of the mode $\mathbf{k}_c$. This result holds for every mode in a stability band and is confirmed by numerical calculations as depicted in Figure 2(a).

Note that in equation (66) $\mathbf{k}_c \neq \mathbf{k}$, since we assumed that $\mathbf{k}$ is in a stability band while $\mathbf{k}_c$ is the central mode of the broadest resonance band of the specific model considered. In a narrow resonance band regime $\mathbf{k}_c$ corresponds to the central mode of the first resonance band. In a broad resonance regime, however, $\mathbf{k}_c$ may not be the central mode, but it is still the mode which has the fastest exponential growth. For instance, in the model studied in [4], $\mathbf{k}_c = 0$ is the mode with the greatest Floquet exponent.

To verify the previous qualitative results we performed a numerical analysis of the problem. Following the notation of Ref. [13], we choose $p(t) = \lambda\cos(2t)$, where λ is a constant, and the noise given in the form $q_{\mathbf{k}}(t) = g\,\lambda\,n(t)\,m(k)$. Here, g is a positive constant, $n(t)$ is a random function of time with characteristic time $T = \Gamma_t^{-1}$ and amplitude equal to unity, and $m(k)$, also a random function with characteristic rate Γ_k and amplitude 1, depends only upon the magnitude of $\mathbf{k}$. We use the dimensionless time t ($=\omega t$) so that the evolution equation for χ_k reads

$$\ddot{\chi}_k + \left[E_k^2 + \lambda \cos(2t)\right] \chi_k = \lambda\, g\, n(t) \sum_{k'} \chi_{k'} m_{k'-k} \,, \tag{67}$$

where λ and E_k^2 are normalized with respect to the frequency ω of the inflaton field. Let us stress that the amplitude of the noise was chosen to be very small, in such a way that all the corresponding homogeneous samples of the noise would not change the solution significantly. This choice assures that the effects we are about to show are consequence solely of the inhomogeneity of the noise.

Figure 2 shows a particular case of broad resonance where E_k^2 is of the form $E_k^2 = \lambda + k^2$ (k is also in units of ω), with $E_o^2 = \lambda = 30$. In this particular case, m_k is chosen to be a smooth function of k which helps us to show that the growth of modes outside a resonance band does not depend upon their initial conditions (see below).

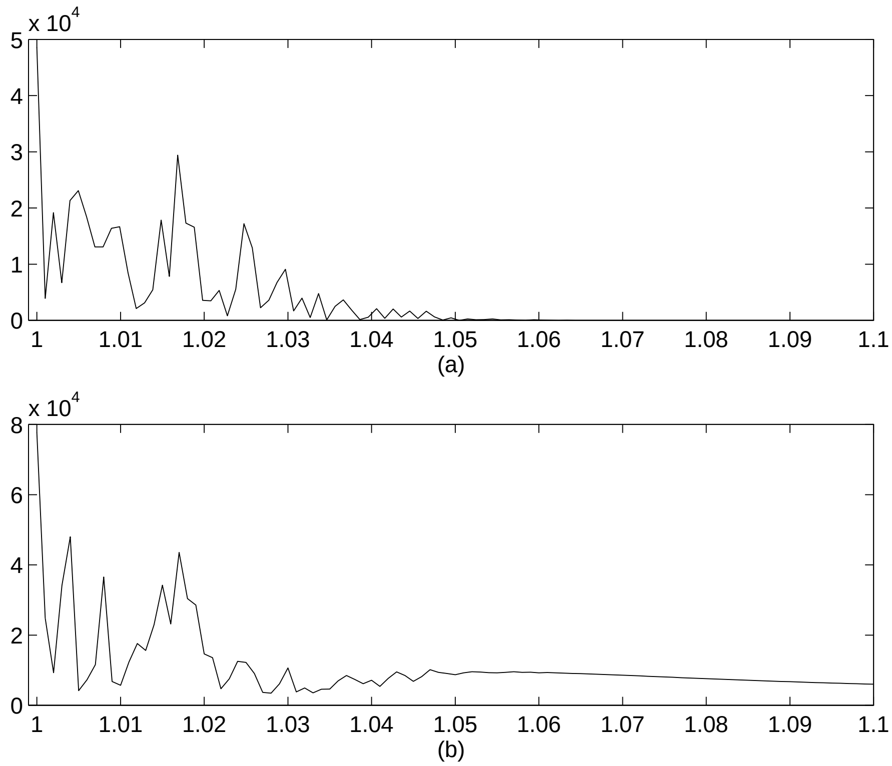

FIG. 2. The maximum amplitude of $\chi_k(t)$ in $t \in [0, 40]$ is plotted (vertical axis) against E_k^2/λ (horizontal axis), $E_o^2 = \lambda = 30$.
(a) No noise is present, $g = 0$.
(b) The noise is characterized by $\Gamma_t = 10$ and $g = 0.3$ (small amplitude and large noise frequency).

This figure shows clearly the stimulated resonance of the modes in a stability band. Fig. 2(a) shows a particular case of broad resonance in the absence of noise. The random peaks arise because we have chosen random initial conditions for both $\chi_k(t)$ and $\dot{\chi}_k(t)$ at $t = 0$. In Fig 2(a) the ratio between the amplitudes of the modes in the stability band (for $E_k^2/\lambda > 1.05$) and the central mode is $O(10^{-4})$, while in Fig 2(b), which shows the results when inhomogeneous noise is present, that ratio is $O(10^{-1})$. Observe that in Fig 2(b) all modes outside the resonance band have almost

the same amplitude. The absence of peaks in that region demonstrates the independence of the amplitudes on the initial conditions of such modes. Those amplitudes depend strongly on the initial conditions and Floquet exponent of the central mode $\chi_{k_c}(t)$. Only 100 modes were taken into account in this example.

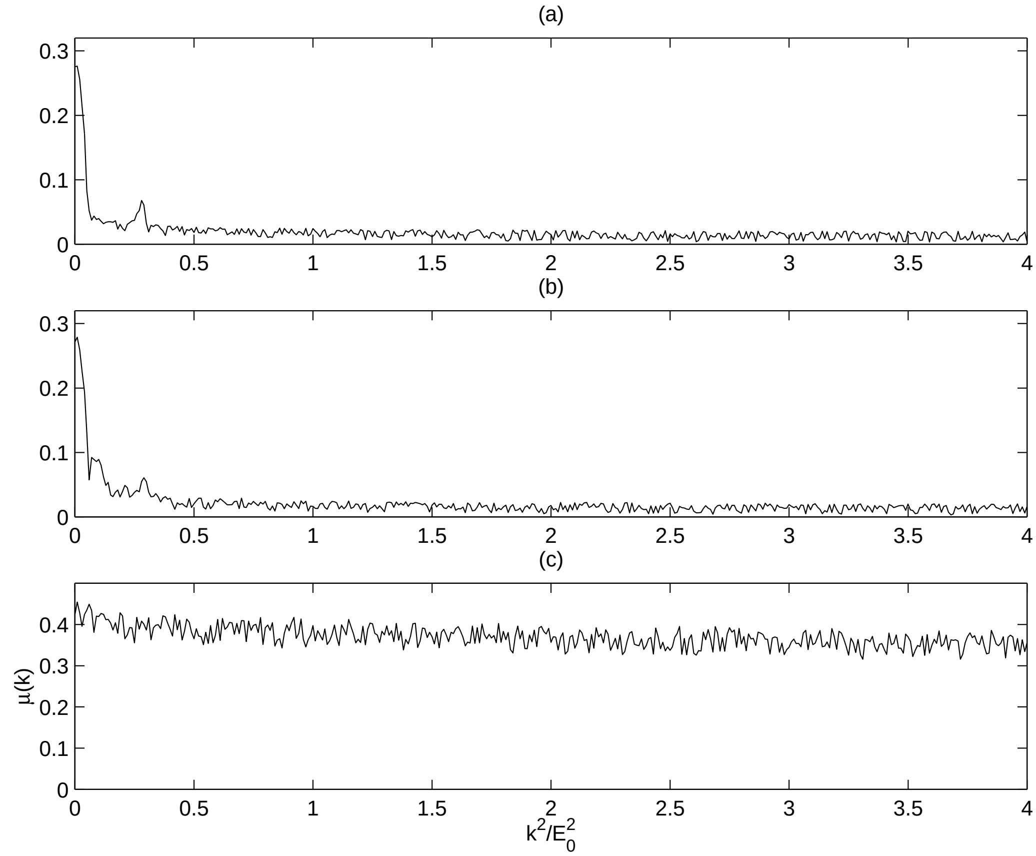

FIG. 3. The Floquet index $\mu(k)$ is plotted (vertical axis) against k^2/E_0^2 (horizontal axis), $E_o^2 = \lambda = 30$.
(a) No noise is present, $g = 0$.
(b) The noise is characterized by $\Gamma_t = 10$ and $g = 0.1$ The noise is homogeneous ($m_{k'-k} = \delta(k' - k)$).
(c) Same parameters as in (b). The noise is inhomogeneous.

Figures 3(a)–3(c) show even more explicitly the effects of inhomogeneities in the noise. In this case k is normalized in such a way that it assumes 400 different values between $k_0 = 0$ end $k_{max} = 2E_0$. In 3(a) no noise is present and we can identify the first resonance band. Figure 3(b) shows the result when we introduced homogeneous noise. The noise is random in both t and k and the amplitude is small, $g = 0.1$. Very small effects due to the noise are observed. Only modes within the resonance band grow exponentially, exactly as in the case of Fig. 3(a). On the other hand, in Fig. 3(c) we observe a very distinct behavior of the modes outside the resonance gap. All the modes have explosive growth, and we cannot distinguish the stability region from the resonance band anymore. The inhomogeneity of the noise has large effects even in the case studied here, where the amplitude of the noise is small compared to the amplitude of the potential $p(t)$.

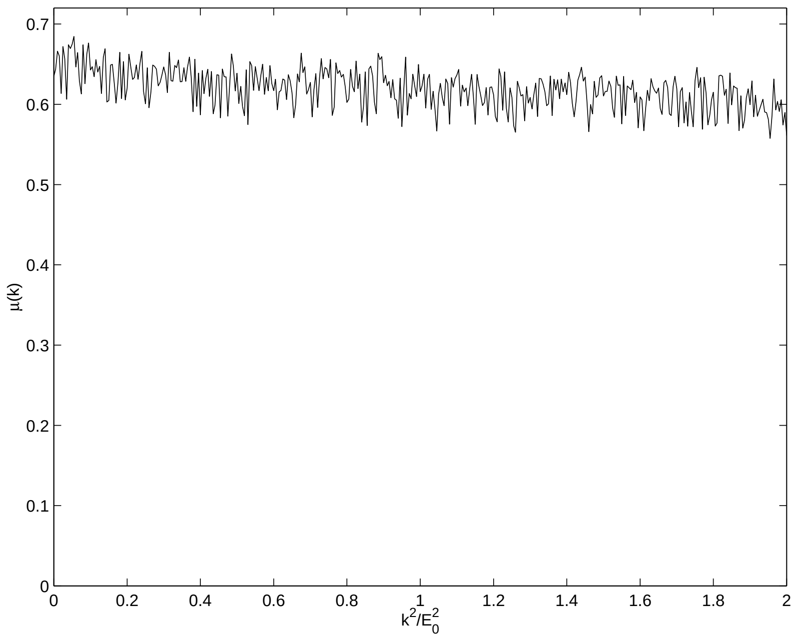

FIG. 4. The same as figure 3(c), but here with Δk twice smaller than in 3(c).

Another interesting point is related to the abovementioned spacing Δk between two successive modes. We can easily see how the Floquet exponent depends on Δk by comparing Fig. 3(c) with Fig. 4. In the last case we used the same parameters as in 3(c), only the normalization of k is different: 400 modes are placed between $k_{min} = 0$ and $k_{max} = \sqrt{2}E_0$. $m_{k'-k}$ was chosen to be a symmetric (otherwise random) matrix in k and k' in both the cases of Fig. 3(c) and Fig. 4. The Floquet exponent increases when the number of Fourier modes is increased. This can be understood since it corresponds to decreasing Δk between two neighboring modes. As a result, we are adding the contribution of a larger number of very close modes which are inside the resonance band, all of them contributing with exponentially growing amplitudes, on the right hand side of Eq. (67).

VI. CONCLUSIONS

In this paper we have shown (modulo some technical points which will be demonstrated in a companion paper) that spatially inhomogeneous noise in the oscillating field which induces the parametric resonance instability does not decrease the Floquet exponent of the instability. This extends the results of [13] where it was shown that homogeneous noise leads to an increase in the Floquet exponent for each Fourier mode. Crucial to our proof was the truncation of the dynamical system to a finite dimensional one by means of infrared and ultraviolet cutoffs, use of the Furstenberg

Theorem concerning the Lyapunov exponent of products of identically distributed random matrices, and the use of compactness arguments to carry over the main results as the cutoffs are removed.

We have also provided numerical evidence (backed up with approximate analytical calculations) which indicate that a much stronger result is true, namely that inhomogeneous noise will spread the exponential growth of the most resonant mode of the system without noise to all of the Fourier modes. An interesting challenge is to provide a mathematically rigorous proof of this stronger result.

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